The scalar triple product

The scalar product and the vector product may be combined into the **scalar triple product** (or mixed product):

$$([\boldsymbol{a},\boldsymbol{b}],\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}\ .$$

Theorem: Given three vectors $\boldsymbol{a}=\{a_x,a_y,a_z\}$, $\boldsymbol{b}=\{b_x,b_y,b_z\}$ and $\boldsymbol{c}=\{c_x,c_y,c_z\}$ in some rectangular coordinate system, the scalar triple product is defined by the formula

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \qquad (9)$$

Proof: Carrying out the scalar product of the vectors

$$\boldsymbol{a} \times \boldsymbol{b} = (a_y b_z - a_z b_y)\boldsymbol{i} + (a_z b_x - a_x b_x)\boldsymbol{j} + (a_x b_y - a_y b_x)\boldsymbol{k}$$

and

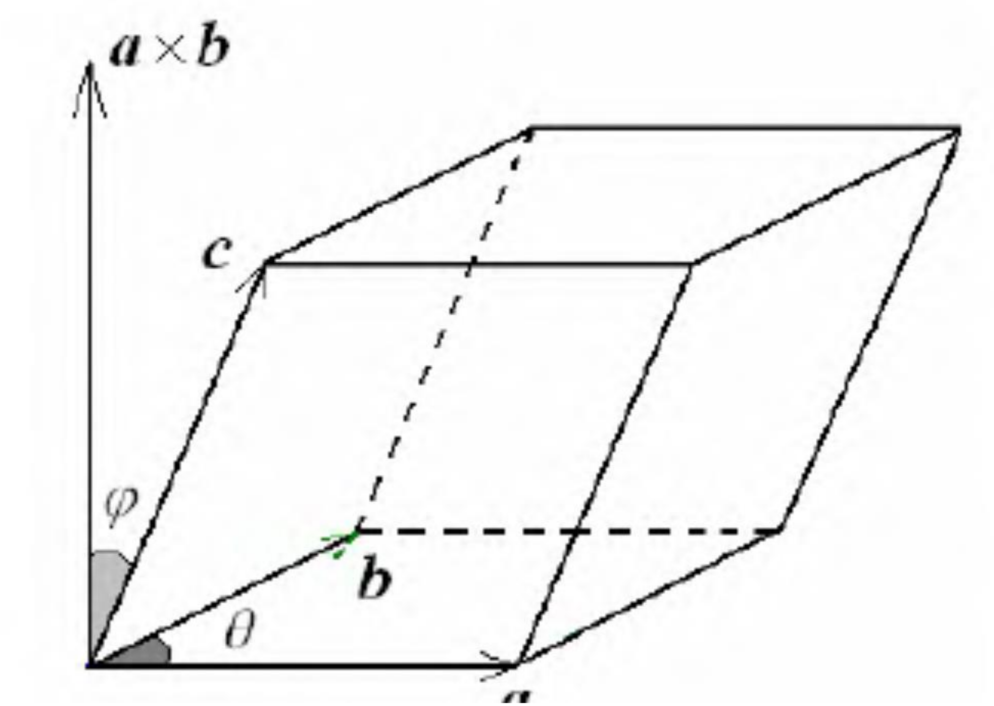
$$\boldsymbol{c} = c_x \boldsymbol{i} + c_y \boldsymbol{j} + c_z \boldsymbol{k}$$

we obtain

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot c = (a_y b_z - a_z b_y) c_x + (a_z b_x - a_x b_x) c_y + (a_x b_y - a_y b_x) c_z$$

$$= \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

Geometric Interpretation. The absolute value of the number $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$ is the volume of a parallelepiped formed by the vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ as it is shown in the figure below.



By the theorem of scalar product,

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = |\boldsymbol{a}\times\boldsymbol{b}|\cdot|\boldsymbol{c}|\cos\varphi .$$

The quantity $|\boldsymbol{a}\times\boldsymbol{b}|$ equals the area of the parallelogram, and the product $|\boldsymbol{c}|\cos\varphi$ equals the height of the parallelepiped.

Corollary 1: If three vectors are coplanar then the scalar triple product is equal to zero.

Corollary 2: Four points *A*, *B*, *C*, and *D* lie in the same plane, if the scalar triple product $(\overrightarrow{AB}\times\overrightarrow{AC})\,\overrightarrow{AD}$ is equal to zero.

Consider the scalar triple product $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$.

1) By the properties of the scalar product, $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{b}\times\boldsymbol{c})\cdot\boldsymbol{a}$.
2) In view of the properties of determinants,

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{vmatrix} b_x & b_y & b_z \\ c_x & c_y & c_z \\ a_x & a_y & a_z \end{vmatrix}.$$

Therefore, $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$.

Since the order of the dot and cross symbols is meaningless, the product $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ is simply denoted by $\boldsymbol{abc}$.

Using the properties of determinants it is not difficult to see that

$$\boldsymbol{abc} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{vmatrix} c_x & c_y & c_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} b_x & b_y & b_z \\ c_x & c_y & c_z \\ a_x & a_y & a_z \end{vmatrix}.$$

Therefore,

$$\boldsymbol{abc} = \boldsymbol{cab} = \boldsymbol{bca}.$$

Likewise,

$$\boldsymbol{abc} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = -\begin{vmatrix} b_x & b_y & b_z \\ a_x & a_y & a_z \\ c_x & c_y & c_z \end{vmatrix} = -\begin{vmatrix} a_x & a_y & a_z \\ c_x & c_y & c_z \\ b_x & b_y & b_z \end{vmatrix},$$

and so

$$\boldsymbol{abc} = -\,\boldsymbol{bac} = -\,\boldsymbol{acb}.$$

In view of the theorem of linear dependent vectors, any three linear dependent vectors are coplanar. Hence,

The triple product of non-zero vectors equals zero, if and only if the vectors are linear dependent.

1) Determine whether the points $A(-1, 2, 2)$, $B(3, 3, 4)$, $C(2, -2, 10)$, and $D(0, 2, 2)$ lie on the same plane.

Solution: Join the point *A* with the other points to obtain the vectors

$$\boldsymbol{a} = \overrightarrow{AB} = \{4, 1, 2\}, \qquad \boldsymbol{b} = \overrightarrow{AC} = \{3, 4, 8\}, \quad \text{and} \qquad \boldsymbol{c} = \overrightarrow{AD} = \{1, 0, 0\}.$$

Find the scalar triple product:

$$\boldsymbol{abc} = \begin{vmatrix} 4 & 1 & 2 \\ 3 & 4 & 8 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} = 0.$$

Therefore, the vectors lie in a plane, that means the given points lie in the same plane.

2) Find the volume V of the tetrahedron with the vertices at the points $A(1,0,2)$, $B(3,-1,4)$, $C(1,5,2)$, and $D(4,4,4)$.

Solution: Consider a parallelepiped whose adjacent vertices are at the given points.

The volume V_p of the parallelepiped is equal to the absolute value of the triple scalar product of the vectors $\vec{AB}$, $\vec{AC}$, and $\vec{AD}$.

The volume of the tetrahedron is given by the formula $V = \frac{1}{6}V_p$.

Since

$$\vec{AB} = \{2, -1, 2\}, \quad \vec{AC} = \{0, 5, 0\}, \quad \text{and} \quad \vec{AD} = \{3, 4, 2\},$$

we obtain

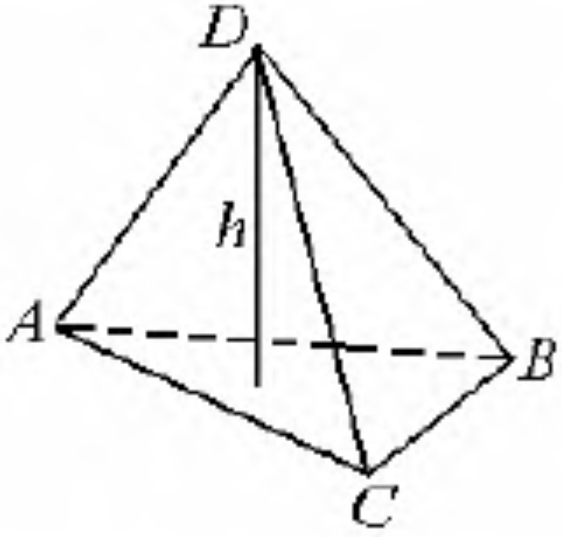
$$\vec{AB}\,\vec{AC}\,\vec{AD} = \begin{vmatrix} 2 & -1 & 2 \\ 0 & 5 & 0 \\ 3 & 4 & 2 \end{vmatrix} = 5\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = 10.$$

Therefore,

$$V = \frac{10}{6} = \frac{5}{3}.$$

3) The tetrahedron is given by the vertices $A(1, 0, 2)$, $B(3, -1, 4)$, $C(1, 5, 2)$, and $D(4, 4, 4)$.

Find the height from the point D to the base ABC.



Solution: In view of the formula

$$V = \frac{1}{3} S \cdot h,$$

where h is the height from the point D, we need to know the volume V of the tetrahedron and the area S of the base ABC to find h.

According to Example 2, the volume of the tetrahedron equals $5/3$.

The area of the triangle ABC can be found just in a similar way as in Example 2, section 1.5.2:

$$A = \frac{1}{2} | \vec{AB} \times \vec{AC} |,$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & 2 \\ 0 & 5 & 0 \end{vmatrix} = 10\boldsymbol{i} - 10\boldsymbol{k}, \quad | \vec{AB} \times \vec{AC} | = 10\sqrt{2}.$$

Therefore,

$$h = \frac{3V}{A} = \frac{5}{5\sqrt{2}} = \frac{\sqrt{2}}{2}.$$